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On algebraic models of relativistic scattering

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Abstract

In this paper we develop an algebraic technique for building relativistic models in the framework of the direct-interaction theories. The interacting mass operator M in the Bakamjian–Thomas construction is related to a quadratic Casimir operator C of a non-compact group G . As a consequence, the S matrix can be gained from an intertwining relation between Weyl-equivalent representations of G . The method is illustrated by explicit application to a model with $SO(3, 1)$ dynamical symmetry.

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1. Introduction

The role of dynamical symmetries in quantum theory has been emphasized by many authors and its implications for physically relevant quantities have been widely recognized. In this connection the importance and relevance of dynamical symmetries has been discussed in several directions [1–3].

Since the work of Zwanziger [4] it has become clear that group-theoretical methods can be successfully applied to the solution of scattering problems. In that paper Zwanziger has shown how the symmetry group $SO(3, 1)$ allows for an algebraic determination of the Coulomb S -matrix elements. A fundamental step towards a deep understanding of scattering problems in a group-theoretical framework was made by the Yale group and others [5] with their method of Euclidean connection. The key point in this major development lies in the observation that the dynamical group G that describes the scattering system in the presence of interactions can be obtained by the deformation [6] of the group G_0 (called asymptotic group) describing the system in the absence of interactions. It appears that knowledge of the interrelation between the representations of G with those of G_0 allows purely algebraic calculations of S -matrix elements for systems whose Hamiltonian H (in the centre of mass system) belongs to the centre of the enveloping algebra of G , i.e.,

$$H = f(C), \tag{1}$$

where C is the Casimir operator of G . Later on [7], it has been argued that the S matrix for systems under consideration is associated with intertwining operators between Weyl-equivalent representations of G (see below). At this stage we note that the operator A is said to intertwine the representation T^χ and $T^{\bar{\chi}}$ of the group G if the relation

$$AT^\chi(g) = T^{\bar{\chi}}(g)A, \quad \text{for all } g \in G \quad (2)$$

or equivalently,

$$\text{Ad}T^\chi(a) = dT^{\bar{\chi}}(a)A, \quad \text{for all } a \in \mathfrak{g} \quad (3)$$

holds, where dT^χ and $dT^{\bar{\chi}}$ are the corresponding representations of the algebra \mathfrak{g} of G [8, 9].

The hypothesis that scattering systems can be completely described by some dynamical group has been verified for almost all interesting non-relativistic problems. Moreover, the algebraic approach is useful not only for systems with exact symmetry, but also for systems with broken symmetry. In this case the arguments in an expression of the S matrix with an exact symmetry are substituted by generic functions of scattering variables, called algebraic potentials [10, 11].

Contrary to the non-relativistic case, the group-theoretical approach to relativistic scattering has not been exploited yet, with the main exception of scattering of a Dirac particle in a Coulomb potential [13], or Coulomb plus scalar potentials [14]. In their study the authors use relativistic wave equations. The algebraic approach, however, is more general, since it relies on a symmetry and does not make any explicit reference to an equation of motion.

Interestingly, there exists an alternative approach to the relativistic particle dynamics based on the work of Bakamjian and Thomas [15] which has the advantage of being somewhat group-theoretical. The point is that one can consider the problem of construction of relativistic theories as that of construction of unitary representations of the inhomogeneous Lorentz group, $ISO(3, 1)$, also known as Poincaré group \mathcal{P} [16].

The Lie algebra of the Poincaré group has ten basis elements, which can be chosen as H , \mathbf{P} , \mathbf{J} and \mathbf{K} , which are the generators of time translations, space translations, space rotations and pure Lorentz transformations, respectively. They satisfy the commutation relations

$$[P_i, P_j] = 0, \quad [P_i, H] = 0, \quad [J_i, H] = 0 \quad (4)$$

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \quad [K_i, K_j] = -i\epsilon_{ijk}J_k, \quad [J_i, K_j] = i\epsilon_{ijk}K_k \quad (5)$$

$$[J_i, P_j] = i\epsilon_{ijk}P_k, \quad [P_i, K_j] = -i\delta_{ij}H. \quad (6)$$

(Throughout this paper units are used in which $\hbar = c = 1$.) Here δ_{ij} is the Kronecker symbol, ϵ_{ijk} the Levi–Civita symbol and the summation convention on repeated indices is assumed. The operators H , \mathbf{P} and \mathbf{J} have the physical significance of energy, momentum and angular momentum. According to [16] an elementary particle should be described by positive energy unitary irreducible representation $(m, s, +)$ of the Poincaré group \mathcal{P} , where m is the mass and s denotes the spin and $+$ means positive energy. Therefore, the description of N noninteracting particles is given by the tensor product of representations $(m_i, s_i, +)$, $i = 1, 2, \dots$

The problem of adding interactions to the noninteracting representation of the Poincaré group \mathcal{P} consistent with the commutation relations (4) has been discussed by Dirac [17]. Although Dirac did not propose a practical method of constructing an interacting representation of the Poincaré group \mathcal{P} , he emphasized that there are three possible schemes for incorporating interactions into the noninteracting representation. These schemes are now called ‘instant form’, ‘front form’ and ‘point form’. Later on, Bakamjian and Thomas [15] have proposed a method for adding interactions to a noninteracting representation of the Poincaré group. In their approach a set of 10 auxiliary operators is introduced that satisfies simple commutation

relations. For example, in the instant form the 10 auxiliary operators are $\{\mathbf{P}, \mathbf{S}, \mathbf{X}, M\}$ with commutation relations

$$[P_i, X_j] = -i\delta_{ij}, \quad [S_i, S_j] = i\epsilon_{ijk} S_k \quad (7)$$

all other commutators vanishing, where \mathbf{S} is the intrinsic spin, $\mathbf{X} = i\nabla_P$, and M is the invariant mass operator. The Poincaré generators are then expressed in terms of $\{\mathbf{P}, \mathbf{S}, \mathbf{X}, M\}$ according to

$$H = \sqrt{M^2 + \mathbf{P}^2}, \quad \mathbf{J} = \mathbf{X} \times \mathbf{P} + \mathbf{S}, \quad \mathbf{K} = \frac{1}{2}(\mathbf{X}\mathbf{H} + \mathbf{H}\mathbf{X}) + \frac{\mathbf{P} \times \mathbf{S}}{H + M}. \quad (8)$$

In the Bakamjian–Thomas approach interactions are added to the mass operator M , while leaving the other nine operators equal to those of the noninteracting system (for review see [18]). As a result, the problem is reduced to an eigenvalue equation for the mass operator M .

The question naturally arises: can a group structure be introduced into the space on which the mass operator M is defined, and if so, does it also have useful consequences? Here the mass operator M is assumed to be a function of the Casimir operator of a non-compact group. This allows pure group-theoretical description of the S matrix. We apply this construction to a scattering system with $SO(3, 1)$ dynamical symmetry.

2. Two-body systems

We consider a system of two interacting spinless particles of mass m_1 and m_2 . In building up a relevant representation of the Poincaré group \mathcal{P} , it is convenient to start with the free system

$$H_0 = \sum_{a=1}^2 h_a, \quad \hat{\mathbf{P}}_0 = \sum_{a=1}^2 \hat{\mathbf{p}}_a, \quad \mathbf{J}_0 = \sum_{a=1}^2 (\hat{\mathbf{x}}_a \times \hat{\mathbf{p}}_a) \quad (9)$$

$$\mathbf{K}_0 = \sum_{a=1}^2 \frac{1}{2} (\hat{\mathbf{x}}_a h_a + h_a \hat{\mathbf{x}}_a), \quad (10)$$

where

$$h_a = \sqrt{m_a^2 + \hat{\mathbf{p}}_a^2}. \quad (11)$$

The operator $\hat{\mathbf{x}}_a$ is canonically conjugate to $\hat{\mathbf{p}}_a$

$$[\hat{p}_a^i, \hat{x}_a^j] = -i\delta^{ij} \quad (12)$$

The non-interacting mass operator \hat{M}_0 is defined by

$$\hat{M}_0^2 = H_0^2 - \hat{\mathbf{P}}_0^2. \quad (13)$$

It commutes with all generators of the Poincaré group \mathcal{P} .

The basis states of the carrier space of this representation can be taken as the tensor products of the single-particle states. They are defined by

$$\hat{\mathbf{p}}_a |\mathbf{p}_1 \mathbf{p}_2\rangle \equiv \mathbf{p}_a |\mathbf{p}_1 \mathbf{p}_2\rangle, \quad a = 1, 2 \quad (14)$$

and normalized so that

$$\langle \mathbf{p}_1 \mathbf{p}_2 | \cdot \mathbf{p}'_1 \mathbf{p}'_2 \rangle = \delta^3(\mathbf{p}_1 - \mathbf{p}'_1) \delta^3(\mathbf{p}_2 - \mathbf{p}'_2) \quad (15)$$

and

$$\int d\mathbf{p}_1 d\mathbf{p}_2 |\mathbf{p}_1 \mathbf{p}_2\rangle \langle \mathbf{p}_1 \mathbf{p}_2| = 1. \quad (16)$$

It will be convenient to make a change of variables from \mathbf{p}_1 and \mathbf{p}_2 to \mathbf{P} and \mathbf{k} , with \mathbf{P} the total momentum and \mathbf{k} the relative momentum. These variables are related to \mathbf{p}_1 and \mathbf{p}_2 by equations [19]

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2, \quad \mathbf{k} = (\varepsilon_2 \mathbf{p}_1 - \varepsilon_1 \mathbf{p}_2) / (\varepsilon_1 + \varepsilon_2) \quad (17)$$

with

$$\varepsilon_a = \frac{1}{2}[E_a + w_a], \quad (18)$$

where E_a and w_a are given by

$$E_a = E_a(\mathbf{p}_a) = \sqrt{m_a^2 + \mathbf{p}_a^2}, \quad w_a = w_a(\mathbf{k}) = \sqrt{m_a^2 + \mathbf{k}^2}. \quad (19)$$

Note that the relative momentum \mathbf{k} is equal to the three momentum of particle 1 in the centre of mass system ($\mathbf{P} = 0$). Hence, w_a is the c.m. energy of a particle of mass m_a . The total c.m. energy can be expressed in the Poincaré-invariant form

$$w_1(\mathbf{k}) + w_2(\mathbf{k}) = \sqrt{s}, \quad (20)$$

where

$$s = (p_1 + p_2)^2 = (E_1 + E_2)^2 - (\mathbf{p}_1 + \mathbf{p}_2)^2. \quad (21)$$

The state vectors $|\mathbf{P}\mathbf{k}\rangle$ and $|\mathbf{p}_1\mathbf{p}_2\rangle$ are related to each other via [19]

$$|\mathbf{P}\mathbf{k}\rangle = [J(\mathbf{p}_1, \mathbf{p}_2)]^{\frac{1}{2}} |\mathbf{p}_1\mathbf{p}_2\rangle, \quad (22)$$

where the Jacobian $J(\mathbf{p}_1, \mathbf{p}_2)$ is given by

$$J(\mathbf{p}_1, \mathbf{p}_2) = \left| \frac{\partial(\mathbf{p}_1\mathbf{p}_2)}{\partial(\mathbf{P}, \mathbf{k})} \right| = \frac{E_1 E_2}{E_1 + E_2} \frac{w_1 + w_2}{w_1 w_2}. \quad (23)$$

On the basis $\{|\mathbf{P}\mathbf{k}\rangle\}$, the non-interacting Hamiltonian H_0 and the non-interacting mass operator \hat{M}_0 are multiplication operators

$$H_0|\mathbf{P}\mathbf{k}\rangle = (w^2 + \mathbf{P}^2)^{1/2}|\mathbf{P}\mathbf{k}\rangle, \quad \hat{M}_0|\mathbf{P}\mathbf{k}\rangle = w|\mathbf{P}\mathbf{k}\rangle, \quad (24)$$

where

$$w = w(\mathbf{k}) = \sqrt{m_1^2 + \mathbf{k}^2} + \sqrt{m_2^2 + \mathbf{k}^2}. \quad (25)$$

Other generators are

$$\mathbf{J}_0 = \hat{\mathbf{X}} \times \hat{\mathbf{P}} + \hat{\mathbf{I}}, \quad \mathbf{K}_0 = \frac{1}{2}(\hat{\mathbf{X}}H_0 + \hat{\mathbf{X}}H_0) - \frac{\hat{\mathbf{I}} \times \hat{\mathbf{P}}}{M_0 + H_0}, \quad (26)$$

where

$$\hat{\mathbf{I}} = \hat{\rho} \times \hat{\mathbf{k}} \quad (27)$$

is the internal angular momentum operator. The operators $\hat{\rho}$ and $\hat{\mathbf{X}}$ are canonically conjugate to $\hat{\mathbf{k}}$ and $\hat{\mathbf{P}}$ and therefore

$$[\hat{k}_i, \hat{\rho}_j] = -i\delta_{ij}, \quad (28)$$

and

$$[\hat{P}_i, \hat{X}_j] = -i\delta_{ij}. \quad (29)$$

To introduce the interaction one lets $M_0 \rightarrow M$, where the interacting mass operator \hat{M} is assumed to be the sum of the non-interacting mass operator \hat{M}_0 plus the mass-operator interaction \hat{V}

$$\hat{M} = \hat{M}_0 + \hat{V}. \quad (30)$$

The set of operators $H, \mathbf{P}, \mathbf{J}$ and \mathbf{K} will satisfy the commutation relations of the Poincaré group \mathcal{P} provided that the following conditions for \hat{V} are satisfied

$$[\hat{V}, \mathbf{P}] = 0, \quad [\hat{V}, \mathbf{J}] = 0. \quad (31)$$

(These constraints lead to the conservation of linear and angular momenta for the interacting system.) In [15] the operator \hat{V} is taken to be a (rotationally) scalar operator function of $\hat{\mathbf{k}}$ and $\hat{\rho}$ only

$$\hat{V} = V(\hat{\mathbf{k}}, \hat{\rho}). \quad (32)$$

The scattering theory within the framework of the Bakamjian–Thomas construction has been considered by the several authors [18–23]. The ‘in’ and ‘out’ scattering states Ψ^\pm are solutions of the relativistic Schrödinger equation

$$H\Psi_{\mathbf{p}_1\mathbf{p}_2}^\pm = [E_1(\mathbf{p}_1) + E_2(\mathbf{p}_2)]\Psi_{\mathbf{p}_1\mathbf{p}_2}^\pm. \quad (33)$$

Although not needed here, we note that the states $\Psi_{\mathbf{p}_1\mathbf{p}_2}^+$ and $\Psi_{\mathbf{p}_1\mathbf{p}_2}^-$ are the solutions of the Lippmann–Schwinger equation

$$\begin{aligned} \Psi_{\mathbf{p}_1\mathbf{p}_2}^\pm &= |\mathbf{p}_1\mathbf{p}_2\rangle + \frac{1}{E_1(\mathbf{p}_1) + E_2(\mathbf{p}_2) - H_0 \pm i0_+} H' \Psi_{\mathbf{p}_1\mathbf{p}_2}^\pm \\ &= |\mathbf{p}_1\mathbf{p}_2\rangle + \frac{1}{E_1(\mathbf{p}_1) + E_2(\mathbf{p}_2) - H \pm i0_+} H' |\mathbf{p}_1\mathbf{p}_2\rangle, \end{aligned} \quad (34)$$

where H' is the interaction Hamiltonian

$$H' = H - H_0 \quad (35)$$

while $|\mathbf{p}_1\mathbf{p}_2\rangle$ is a solution of

$$H_0|\mathbf{p}_1\mathbf{p}_2\rangle = [E_1(\mathbf{p}_1) + E_2(\mathbf{p}_2)]|\mathbf{p}_1\mathbf{p}_2\rangle. \quad (36)$$

The scattering operator S is defined by [24, 25]

$$\Psi_{\mathbf{p}_1\mathbf{p}_2}^+ = \hat{S}\Psi_{\mathbf{p}_1\mathbf{p}_2}^- \quad (37)$$

and the S-matrix elements are accordingly determined by

$$S(\mathbf{p}'_1, \mathbf{p}'_2; \mathbf{p}_1, \mathbf{p}_2) = \langle \Psi_{\mathbf{p}'_1\mathbf{p}'_2}^+ | \hat{S} | \Psi_{\mathbf{p}_1\mathbf{p}_2}^- \rangle = \langle \Psi_{\mathbf{p}'_1\mathbf{p}'_2}^- | \Psi_{\mathbf{p}_1\mathbf{p}_2}^+ \rangle. \quad (38)$$

It has been proved (e.g. section 6 of [19]) that the Bakamjian–Thomas construction guarantees the Poincaré invariance of the operator S .

In [19] has been shown that

$$S(\mathbf{p}'_1, \mathbf{p}'_2; \mathbf{p}_1, \mathbf{p}_2) = [J(\mathbf{p}'_1, \mathbf{p}'_2)J(\mathbf{p}_1, \mathbf{p}_2)]^{-\frac{1}{2}} \delta^3(\mathbf{P}' - \mathbf{P}) \langle \Phi_{\mathbf{k}'}^- | \Phi_{\mathbf{k}}^+ \rangle, \quad (39)$$

where $\Phi_{\mathbf{k}}^\pm$ denote the ‘in’ and ‘out’ eigenstate of the mass operator \hat{M} , with asymptotic relative momentum \mathbf{k}

$$\hat{M}\Phi_{\mathbf{k}}^\pm = w\Phi_{\mathbf{k}}^\pm, \quad (40)$$

where the c.m. energy w is given by

$$w = w(\mathbf{k}) = \sqrt{m_1^2 + \mathbf{k}^2} + \sqrt{m_2^2 + \mathbf{k}^2}. \quad (41)$$

More precisely, the states $\Phi_{\mathbf{k}}^\pm$ are the solutions of the Lippmann–Schwinger equation

$$\Phi_{\mathbf{k}}^\pm = |\mathbf{k}\rangle + \frac{1}{w(\mathbf{k}) - \hat{M}_0 \pm i0_+} \hat{V} \Phi_{\mathbf{k}}^\pm \quad (42)$$

$$= |\mathbf{k}\rangle + \frac{1}{w(\mathbf{k}) - \hat{M} \pm i0_+} \hat{V} |\mathbf{k}\rangle, \quad (43)$$

where $|\mathbf{k}\rangle$ is a solution of

$$\hat{M}_0|\mathbf{k}\rangle = w|\mathbf{k}\rangle. \quad (44)$$

For two particles with equal masses $m_1 = m_2 = m$ equation (40) simplifies to

$$(2\sqrt{m^2 + \hat{\mathbf{k}}^2} + \hat{V})\Phi_{\mathbf{k}}^{\pm} = 2\sqrt{m^2 + \mathbf{k}^2}\Phi_{\mathbf{k}}^{\pm}. \quad (45)$$

Squaring both sides and making some rearrangement, equation (45) can put in the form [22, 26, 27]

$$\left(\frac{\hat{\mathbf{k}}^2}{m} + \hat{V}\right)\Phi_{\mathbf{k}}^{\pm} = \mathcal{E}\Phi_{\mathbf{k}}^{\pm} \quad (46)$$

with $\mathcal{E} = \mathbf{k}^2/m$ and

$$\hat{V} = \frac{1}{2m}\{\hat{V}, \sqrt{m^2 + \hat{\mathbf{k}}^2}\} + \frac{\hat{V}^2}{4m}. \quad (47)$$

Equation (46) is identical in structure to a non-relativistic Schrödinger equation.

We can define the S matrix related to (46)

$$\Phi_{\mathbf{k}}^+ = \mathcal{S}\Phi_{\mathbf{k}}^-. \quad (48)$$

Note, this S matrix is different from the previous one. The principal difference between the scattering operators in (37) and (48) is that while the former commutes with generators $H, \mathbf{P}, \mathbf{J}$ and \mathbf{K} of the Poincaré group \mathcal{P} , the latter commutes with generators $\hat{\mathbf{I}}$ of a group being isomorphic to $SO(3)$.

According to (48)

$$\langle \Phi_{\mathbf{k}'}^- | \Phi_{\mathbf{k}}^+ \rangle = \langle \Phi_{\mathbf{k}'}^+ | \mathcal{S}\Phi_{\mathbf{k}}^+ \rangle = \mathcal{S}(\mathbf{k}', \mathbf{k}). \quad (49)$$

Separating from $\mathcal{S}(\mathbf{k}', \mathbf{k})$ the non-interacting part, it is customary to write

$$\mathcal{S}(\mathbf{k}', \mathbf{k}) = \delta^3(\mathbf{k}' - \mathbf{k}) - 2\pi i\delta(\mathcal{E}' - \mathcal{E})\mathcal{T}(\mathbf{k}', \mathbf{k}), \quad (50)$$

where $\mathcal{T}(\mathbf{k}', \mathbf{k})$ is called the T matrix or transition amplitude. Since \mathcal{V} is rotationally invariant the transition amplitude $\mathcal{T}(\mathbf{k}', \mathbf{k})$ may be a function of $k \equiv |\mathbf{k}|$ and $\cos\theta = \mathbf{n}' \cdot \mathbf{n}$ only, where $\mathbf{n}' = \mathbf{k}'/k'$ and $\mathbf{n} = \mathbf{k}/k$. It related to the c.m. scattering amplitude $f(\theta)$ by equation [28]

$$\mathcal{T}(\mathbf{k}', \mathbf{k}) = -\frac{1}{2\pi^2 m} f(\theta), \quad (51)$$

where θ is the c.m. scattering angle.

Inserting these relations into equation (39) and using the identities

$$\delta^3(\mathbf{P}' - \mathbf{P})\delta(\mathcal{E}' - \mathcal{E}) = \frac{2m}{E}\delta^3(\mathbf{P}' - \mathbf{P})\delta(E' - E) \quad (52)$$

and [19]

$$\delta^3(\mathbf{P}' - \mathbf{P})\delta^3(\mathbf{k}' - \mathbf{k}) = J(\mathbf{p}_1, \mathbf{p}_2)\delta^3(\mathbf{p}'_1 - \mathbf{p}_1)\delta^3(\mathbf{p}'_2 - \mathbf{p}_2) \quad (53)$$

we find that

$$\begin{aligned} S(\mathbf{p}'_1, \mathbf{p}'_2; \mathbf{p}_1, \mathbf{p}_2) &= \delta^3(\mathbf{p}'_1 - \mathbf{p}_1)\delta^3(\mathbf{p}'_2 - \mathbf{p}_2) - 2\pi i\delta^4(p'_1 + p'_2 - p_1 - p_2) \\ &\times \frac{\mathcal{M}(p_1, p_2; p'_1, p'_2)}{(2E_1)^{1/2}(2E_2)^{1/2}(2E'_1)^{1/2}(2E'_2)^{1/2}}, \end{aligned} \quad (54)$$

where

$$\mathcal{M} = -\sqrt{s} f(\theta)/\pi^2 \quad (55)$$

is an invariant amplitude. Thus, in order to determine $S(\mathbf{p}'_1, \mathbf{p}'_2; \mathbf{p}_1, \mathbf{p}_2)$, it is sufficient to know the c.m. scattering amplitude $f(\theta)$

$$f(\theta) = \frac{1}{2ik} \sum_{l=0}^{\infty} (2l+1)(S_l - 1)P_l(\cos\theta), \tag{56}$$

where P_l are Legendre polynomials and S_l is the S matrix element for angular momentum l .

The two-body cross section is given by

$$\frac{d\sigma}{dt} = \frac{\pi^5 |\mathcal{M}|^2}{\sqrt{(p_1 p_2)^2 - m^4}}, \tag{57}$$

where t is the momentum transfer squared, i.e.,

$$t = (p'_1 - p_1)^2. \tag{58}$$

It may be written as

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 \tag{59}$$

It seems reasonable to assume that there might be a relativistic interacting system that has a non-compact group G as dynamical symmetry group in the sense that

$$\frac{\hat{\mathbf{k}}^2}{m} + \hat{\mathcal{V}} = f(C), \tag{60}$$

where C is the Casimir operator of G . If that is the case, then the S matrix for systems under consideration is constrained to satisfy [7]

$$ST^\chi(g) = T^{\tilde{\chi}}(g)S, \quad \text{for all } g \in G \tag{61}$$

or equivalently,

$$S dT^\chi(a) = dT^{\tilde{\chi}}(a)S, \quad \text{for all } a \in \mathfrak{g}, \tag{62}$$

where T^χ and $T^{\tilde{\chi}}$ are Weyl-equivalent representations of G specified by labels χ and $\tilde{\chi}$, while dT^χ and $dT^{\tilde{\chi}}$ are the corresponding representations of the algebra \mathfrak{g} of G . (The representations T^χ and $T^{\tilde{\chi}}$ have the same Casimir eigenvalues. Such representations are called Weyl equivalent.) Equations (61) and (62) have much restriction power and are used in deriving the S matrix [12].

In order to avoid misunderstanding, we make a few comments on equation (61) or (62). To start with, it should be pointed out that we have in the subspace of scattering states two complete orthonormal systems, $\{\Phi^+\}$ and $\{\Phi^-\}$. The state vectors Φ^- transform according to the representation $T^\chi(g)$, while the state vectors Φ^+ transform according to $T^{\tilde{\chi}}(g)$. Since by definition the operator S maps each Φ^- on the corresponding Φ^+ , then $T^{\tilde{\chi}}(g)\Phi^+ = ST^\chi(g)\Phi^-$ so that $T^{\tilde{\chi}}(g)S\Phi^- = ST^\chi(g)\Phi^-$. This means that S must satisfy equation (61). Moreover, if S intertwines representations T^χ and $T^{\tilde{\chi}}$ of the Lie group G , it also intertwines the representations dT^χ and $dT^{\tilde{\chi}}$ of the Lie algebra \mathfrak{g} .

Finally, we would like to emphasize that, in general, $\hat{\mathbf{k}}^2/m + \hat{\mathcal{V}}$ is some rotationally invariant operator function of $\hat{\mathbf{k}}$ and \hat{p} . So the geometrical invariance algebra of (46) is the algebra generated by $\hat{\mathbf{I}}$. In other words, the group G has a subgroup being isomorphic to $SO(3)$. Since the representations T^χ and $T^{\tilde{\chi}}$ are identical for compact subgroups of G [29, 30], it follows from (61) and (62) that

$$[T^\chi(g), S] = 0, \quad \text{if } g \in SO(3). \tag{63}$$

or

$$[dT^\chi(a), S] = 0, \quad \text{if } a \in \mathfrak{so}(3). \tag{64}$$

as it should be.

Let us apply this construction to scattering systems that have $SO(3, 1)$ as the dynamical symmetry group, i.e.

$$\frac{\hat{\mathbf{k}}^2}{m} + \hat{\mathcal{V}} = f(C_1). \quad (65)$$

We first note that [31, 32] the internal angular momentum operator $\hat{\mathbf{I}}$ and the operator $\hat{\mathbf{N}}$ defined by

$$\hat{\mathbf{N}} = \frac{1}{2\sqrt{\hat{\mathbf{k}}^2}} [\hat{\mathbf{k}} \times \hat{\mathbf{I}} - \hat{\mathbf{I}} \times \hat{\mathbf{k}}] \quad (66)$$

span the Lie algebra of the Lorentz group $SO(3, 1)$

$$[\hat{l}_i, \hat{l}_j] = i\epsilon_{ijk}\hat{l}_k, \quad [\hat{l}_i, N_j] = i\epsilon_{ijk}N_k, \quad [N_i, N_j] = -i\epsilon_{ijk}\hat{l}_k. \quad (67)$$

(The dynamical algebra (67) should not be confused with Lorentz subalgebra (5) of the Poincaré group.) These commutation relations can be easily calculated by making use of equations (28) and

$$[\hat{l}_i, \hat{k}_j] = i\epsilon_{ijk}\hat{k}_k. \quad (68)$$

Then, the S matrices for such systems can be obtained from equation (61) or (62). To this end, a few facts from representation theory of the group $SO(3, 1)$ are useful.

The unitary irreducible representations of $SO(3, 1)$ are known to form three series: principal, supplementary and discrete. It is also known that only the principal series describes the scattering states. The principal series of $SO(3, 1)$ are characterized by the pair $\chi = (\tau, \lambda)$, where $\lambda = 0, \pm\frac{1}{2}, \pm 1, \dots$, while $-\infty < \tau < \infty$. The representations specified by labels $\chi = (\tau, \lambda)$ and $\tilde{\chi} = (-\tau, -\lambda)$ are Weyl equivalent. In every UIR of principal series of $SO(3, 1)$ the Casimir invariants C_1 and C_2

$$C_1 = \mathbf{I}^2 - \mathbf{N}^2, \quad C_2 = \mathbf{I} \cdot \mathbf{N} \quad (69)$$

become equal to a multiple of the identity operator I

$$C_1 = -(\lambda^2 + \tau^2 + 1)I, \quad C_2 = \lambda\tau I. \quad (70)$$

It is also worth noticing [32] that the second Casimir invariant C_2 is identically zero for the above realization of $SO(3, 1)$

$$C_2 = \frac{1}{2\sqrt{\hat{\mathbf{k}}^2}} [\hat{\mathbf{I}} \cdot (\hat{\mathbf{k}} \times \hat{\mathbf{I}}) - \hat{\mathbf{I}} \cdot (\hat{\mathbf{I}} \times \hat{\mathbf{k}})] = 0. \quad (71)$$

Consequently, the relevant unitary representations will be the principal series representation $(\tau, 0)$. It is worthwhile to point out that the second label, λ , of the (τ, λ) irrep is connected with the helicities of particles; that is why we have the $(\tau, 0)$ irrep for spinless particles.

The representations specified by $\chi = (\tau, 0)$ can be realized in the Hilbert space spanned by the eigenvectors $|\tau 0; l\mu\rangle$ of $\hat{\mathbf{I}}^2$ and \hat{l}_3 . The operators \hat{l}_i, \hat{N}_i are then defined by

$$\begin{aligned} \hat{l}_3^\chi |l\mu\rangle &= \mu |l\mu\rangle \\ \hat{l}_\pm^\chi |l\mu\rangle &= [(l \mp \mu)(l \pm \mu + 1)]^{\frac{1}{2}} |l, \mu \pm 1\rangle \\ \hat{N}_3^\chi |l\mu\rangle &= i(-1 + i\tau - l)a_{l+1, \mu} |l + 1, \mu\rangle + i(i\tau + l)a_{l, \mu} |l - 1, \mu\rangle \\ \hat{N}_\pm^\chi |l\mu\rangle &= \pm i(1 - i\tau + l)b_{l+1, \pm\mu+1} |l + 1, \mu \pm 1\rangle \pm i(i\tau + l)b_{l, \mp\mu} |l - 1, \mu \pm 1\rangle \end{aligned}$$

where $\hat{N}_\pm = \hat{N}_1 \pm i\hat{N}_2, \hat{N}_\pm = \hat{N}_1 \pm i\hat{N}_2, |l\mu\rangle \equiv |\tau 0; l\mu\rangle$ and

$$a_{l, \mu} = \sqrt{\frac{(l + \mu)(l - \mu)}{(2l + 1)(2l - 1)}}, \quad b_{l, \mu} = \sqrt{\frac{(l + \mu)(l + \mu - 1)}{(2l + 1)(2l - 1)}}. \quad (72)$$

We can now evaluate the S matrix from (62). To do this let us write equation (62) for generators \hat{l}_3, \hat{l}_\pm and \hat{N}_3

$$S\hat{l}_3^\chi = \hat{l}_3^\chi S \tag{73}$$

$$S\hat{l}_\pm^\chi = \hat{l}_\pm^\chi S \tag{74}$$

$$S\hat{N}_3^\chi = \hat{N}_3^\chi S. \tag{75}$$

Applying both sides of equations (73) and (74) to the basis vector $|l\mu\rangle$ we find that the S -matrix in the angular momentum representation is diagonal and its matrix elements are independent of μ , i.e.,

$$S|l\mu\rangle = S_l|l\mu\rangle. \tag{76}$$

(Observe that the operator S commutes with all J_i^χ 's, as expected.) The value of its diagonal elements can be defined by using (73). As a result we obtain the recurrence relation

$$(1 - i\tau + l)S_{l+1} = (1 + i\tau + l)S_l, \quad (-i\tau + l)S_l = (i\tau + l)S_{l-1} \tag{77}$$

which implies that

$$S_l = \frac{\Gamma(1 + i\tau + l)}{\Gamma(1 - i\tau + l)}. \tag{78}$$

Inserting this into equation (56) we obtain

$$f(\theta) = \frac{1}{2ik} \frac{\Gamma(1 + i\tau)}{\Gamma(-i\tau)} \frac{1}{\sin^2 \frac{\theta}{2}} \exp \left[-i\tau \ln \left(\sin^2 \frac{\theta}{2} \right) \right], \quad \theta \neq 0, \tag{79}$$

where the momentum-dependent parameter τ is determined by relation (65). (For (78) the expansion diverges as a function, but it converges as a distribution [33].) For example, in analogy to the non-relativistic Coulomb interaction, we can propose

$$\frac{\hat{\mathbf{k}}^2}{m} + \hat{\mathcal{V}} = -\frac{\alpha^2 m}{4(C_1 + 1)}, \tag{80}$$

where α denotes the strength of interaction. This means that

$$\left(\frac{\hat{\mathbf{k}}^2}{m} + \hat{\mathcal{V}} \right) \Phi_{\mathbf{k}}^\pm = -\left(\frac{\alpha^2 m}{4(C_1 + 1)} \right) \Phi_{\mathbf{k}}^\pm. \tag{81}$$

So, taking into account equations (46) and

$$C_1 \Phi_{\mathbf{k}}^\pm = -(\tau^2 + 1) \Phi_{\mathbf{k}}^\pm \tag{82}$$

we have

$$\tau = \frac{\alpha m}{2k}. \tag{83}$$

It then follows that (in ordinary units)

$$f_c(\theta) = \frac{\alpha}{m v^2 \sin^2 \frac{\theta}{2}} (1 - \beta^2) \exp \left[-i\tau \ln \left(\sin^2 \frac{\theta}{2} \right) + i\tau + 2i\eta \right], \quad \theta \neq 0 \tag{84}$$

with $\tau = \frac{\alpha m}{2\hbar k} = \frac{\alpha}{\hbar v} (1 - \beta^2)^{1/2}$. Here $\beta = v/c$, $\eta = \arg \Gamma(1 + i\tau)$ and v is the relative velocity of the particles.

If two spinless particles are identical the indistinguishability of them leads to the c.m. scattering amplitude $f_c^s(\theta)$ of the form

$$f_c^s(\theta) = f_c(\theta) + f_c(\pi - \theta). \tag{85}$$

This results in the differential cross section,

$$\frac{d\sigma}{d\Omega} = |f_c(\theta) + f_c(\pi - \theta)|^2 = \left(\frac{\alpha}{mv^2}\right)^2 \left\{ \frac{1}{\sin^4 \frac{\theta}{2}} + \frac{1}{\cos^4 \frac{\theta}{2}} + \frac{2}{\sin^2 \frac{\theta}{2} \cos^2 \frac{\theta}{2}} \cos \right. \\ \left. \times \left[\frac{\alpha}{\hbar v} (1 - \beta^2)^{1/2} \ln \tan^2 \frac{\theta}{2} \right] \right\} (1 - \beta^2)^2. \quad (86)$$

If we take the non-relativistic limit $\beta \rightarrow 0$, we gain the Mott formula [34] for the Coulomb scattering of two identical spinless bosons.

3. Conclusions and outlook

In this paper we developed an algebraic technique for building two-body relativistic models in the framework of the direct-interaction theories, i.e., theories in which there are no external fields. We have demonstrated how the algebraic technique [7], originally conceived for non-relativistic scattering, can be generalized for the construction of relativistic scattering matrix. Crucial in all the developments was the assumption that the mass operator M for given scattering system is related to the Casimir operator of some non-compact group G . The results described in this paper could be extended in several ways. One of these would be their use for the scattering models with spins. Also, our analysis has been restricted to two-body systems. It would be interesting to generalize the technique discussed in this paper to the study of scattering problems for many-body systems. These and other extensions will be studied in subsequent papers.

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